

The First Eigenvalue for the Bi-Beltrami-Laplacian on Minimal Isoparametric Hypersurfaces of $\mathbb{S}^{n+1}(1)$

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Abstract

In this paper, we investigate the first eigenvalues of two closed eigenvalue problems of the bi-Beltrami-Laplacian on minimal embedded isoparametric hypersurface in the unit sphere $\mathbb{S}^{n+1}(1)$. Although many mathematicians want to derive the corresponding results for the first eigenvalues of bi-Beltrami-Laplacian, they encountered great difficulties in proving the limit theorem of the version of bi-Beltrami-Laplacian along with the strategy due to I. Chavel and E. A. Feldman (Journal of Functional Analysis, 30 (1978), 198-222) and S. Ozawa (Duke Mathematics Journal, 48 (1981), 767-778). Therefore, as the author knows, there are no any results of Tang-Yan type (Journal of Differential Geometry, 94 (2013) 521-540). However, by the variational argument, we overcome the difficulties and determine the first eigenvalues of the bi-Beltrami-Laplacian in the sense of isoparametric hypersurfaces. We note that our proof is quite simple.

Keywords: the first eigenvalue; bi-Beltrami-Laplacian; closed eigenvalue problem; isoparametric hypersurfaces.

2000 MSC 35P15, 53C40.

1 Introduction

Let (M^n, g) be an n -dimensional complete Riemannian manifolds. For any $C^k, k \geq 2$ function f on M^n , define the Beltrami-Laplacian of f by

$$\Delta = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^n \partial_i g^{ji} \sqrt{\det(g)} \partial_j,$$

and then, it is a positive elliptic operator. We consider the following closed eigenvalue problem:

$$\Delta u = \lambda u. \tag{1.1}$$

When M^n is a minimal embedded hypersurface in the unit sphere $\mathbb{S}^{n+1}(1)$, it follows from Takahashi Theorem that $\lambda_1(M)$ is not greater than n . In this connection, S.-T. Yau proposed in 1982 the following conjecture [16, 22]: The first eigenvalue of every closed minimal hypersurface M^n in the unit sphere $\mathbb{S}^{n+1}(1)$ is just the dimension of the hypersurface M^n . Attacking the Yau's conjecture, many mathematicians contributed to this problem, we refer to [4, 7, 9, 16–18, 20] and reference therein. In particular, based on the generic far reaching results on the classification of isoparametric hypersurfaces in $\mathbb{S}^{n+1}(1)$ (we refer to [1, 3, 5, 6, 8, 10, 11, 14, 15] and the references therein), Tang and Yan [20] made a landmark breakthrough to Yau's conjecture. They considered a little more restricted problem of Yau's conjecture for closed minimal isoparametric hypersurfaces M^n in $\mathbb{S}^{n+1}(1)$. In detail, they proved the following:

Theorem (Tang and Yan). *Let M^n be a closed minimal isoparametric hypersurface in the unit sphere $\mathbb{S}^{n+1}(1)$. Then, $\lambda_1(M^n) = n$.*

Up to now, Yau's conjecture is still open.

Next, we consider the following two closed eigenvalue problems of bi-Beltrami-Laplaican:

$$\Delta^2 u = \Lambda u; \quad (1.2)$$

and

$$\Delta^2 u = \Gamma \Delta u. \quad (1.3)$$

Let Λ_i and Γ_i denote the i -th eigenvalue of eigenvalue problem (1.2) and eigenvalue problem (1.3), respectively. Then, the eigenvalues of this eigenvalue problems (1.2) and (1.3) are real and discrete:

$$0 = \Lambda_0 \leq \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \rightarrow +\infty,$$

and

$$0 = \Gamma_0 \leq \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \rightarrow +\infty,$$

where each Λ_i and each Γ_i have finite multiplicity. In [20], Tang and Yan noted that *the calculation of the spectrum of the Laplace-Beltrami operator, even of the first eigenvalue, is rather complicated and difficult*. In fact, the author thinks that, comparing with the case of the Laplace-Beltrami operator, the calculation of the first eigenvalue of the bi-Laplace-Beltrami operator maybe more complicated and difficult. In order to derive the results for the first eigenvalues of the bi-Beltrami-Laplacian on isoparametric hypersurface which is embedded into the $(n + 1)$ -dimensional unit sphere \mathbb{S}^{n+1} , the natural way is to use a series of techniques due to Muto-Ohnita-Urakawa [9], Kotani [7], and Solomon [17, 18], Muto [12] and Tang-Yan [20]. Therefore, firstly, it seems unavoidable to prove a crucial limit theorem in the sense of bi-Beltrami-Laplacian along with the strategy due to I. Chavel and E. A. Feldman [2] and S. Ozawa [13]. In fact, many mathematicians considered the problem, but, technically, they encountered great difficulty in proving the limit theorem. Therefore, as we know, there are no any results of Tang-Yan type in the sense of bi-Laplace-Beltrami operator. However, by an observation of variational principle, we find that it is not necessary to prove the limit theorem but provide an alternative strategy to overcome those difficulties. More precisely, by variational principle and based on Tang-Yan's work [20], we prove that following:

Theorem 1.1. *Let M^n be a closed minimal isoparametric hypersurface in the unit sphere $\mathbb{S}^{n+1}(1)$. Then*

$$\Lambda_1(M^n) = n^2, \text{ and } \Gamma_1(M^n) = n.$$

2 The Proof of Main Result

In this section, we give a very short proof of theorem 1.1. Firstly, we need the following theorem [19]:

Theorem 2.1. (Takahashi) *Let M^n be an n -dimensional Riemannian manifold. For an isometric immersion $\psi : M^n \rightarrow \mathbb{S}^{n+1}$ it is minimal immersion into \mathbb{S}^n if and only if*

$$\Delta\psi = n\psi. \quad (2.1)$$

By the variational principle and Schwarz's inequality, it is easy to prove the following lemma:

Lemma 2.2. *Let λ_i , Λ_i and Γ_i denote the i -th eigenvalues of eigenvalue problem (1.1), eigenvalue problem (1.2) and eigenvalue problem (1.3), respectively. Then,*

$$\Lambda_k \geq \lambda_k^2, \text{ and } \Gamma_k \geq \lambda_k. \quad (2.2)$$

Proof of theorem 1.1. In (2.1), for any $i, (i = 1, 2, \dots, n+2)$, we take the standard coordinate function $\psi = x_i$, and then, one has $\Delta x_i = n x_i$, and thus, $\Delta^2 x_i = n \Delta x_i$. Using Takahashi's theorem again, it follows that $\Delta^2 x_i = n^2 x_i$. Therefore, by the variational principle, we have $\Lambda_1 \leq n^2$ and $\Gamma_1 \leq n$. Furthermore, the inequalities (2.2) in lemma 2.2 implies $\Lambda_1 \geq n^2$ and $\Gamma_1 \geq n$. Hence, we have $\Lambda_1 = n^2$ and $\Gamma_1 = n$. The proof of theorem 1.1 is complete. □

Remark 2.1. By the observation for the proof, it is not difficult to see that we can remove the condition of isoparametric hypersurfaces in theorem 1.1, this is, both $\Lambda_1(M^n) = n^2$ and $\Gamma_1(M^n) = n$ hold without the assumption of the isoparametric condition, if one can prove that Yau's conjecture is true.

Remark 2.2. According to the variational principle and Choi and Wang's result [4], we can easily prove that,

$$\Lambda_1(M^n) \geq \frac{n^2}{4}, \text{ and } \Gamma_1(M^n) \geq \frac{n}{2},$$

without the assumption of isoparametric hypersurfaces.

Remark 2.3. For eigenvalues of the closed eigenvalue problem (1.2) and closed eigenvalue problem (1.3) of the Bi-Laplacian on focal submanifolds of isoparametric hypersurfaces in the unit sphere, we can obtain similar results by making use of the variational principle and the results on the first eigenvalue of Laplacian λ_1 due to Tang and Yan [20] and Tang, Xie and Yan [21].

Acknowledgment. The author is supported by the National Nature Science Foundation of China (Grant No. 11401268).

References

- [1] T. E. Cecil, Q.-S. Chi and G.R. Jensen, Isoparametric hypersurfaces with four principal curvatures, *Annals of mathematics*, 2007, **166**, 1-76.
- [2] I. Chavel and E. A. Feldman, Spectra of domains in compact manifolds, *Journal of Functional Analysis*, 1978, **30**(2): 198-222.
- [3] Q. S. Chi, Isoparametric hypersurfaces with four principal curvatures, *Journal of Differential Geometry*, 2013, **94**(3): 469-504.
- [4] H. I. Choi and A. N. Wang, A first eigenvalue estimate for minimal hypersurfaces, *Journal of differential geometry*, 1983, **18**(3): 559-562.
- [5] J. Dorfmeister and E. Neher, Isoparametric hypersurfaces, case $g = 6, m = 1$, *Communications in Algebra*, 1985, **13**(11): 2299-2368.
- [6] S. Immervoll, On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres, *Annals of mathematics*, 2008: 1011-1024.
- [7] M. Kotani, The first eigenvalue of homogeneous minimal hypersurfaces in a unit sphere $\mathbb{S}^{n+1}(1)$, *Tôhoku Mathematical Journal, Second Series*, 1985, **37**(4): 523-532.
- [8] R. Miyaoka, Isoparametric hypersurfaces with $(g, m) = (6, 2)$, *Annals of Mathematics*, 2013, **177**(1): 53-110.
- [9] H. Muto, Y. Ohnita and H. Urakawa, Homogeneous minimal hypersurfaces in the unit sphere and the first eigenvalue of the Laplacian, *Tôhoku Mathematical Journal, Second Series*, 1984, **36**(2): 253-267.
- [10] H.-F. Münzner, Isoparametrische Hyperflächen in Sphären I, *Mathematische Annalen*, 1980, **251**(1): 57-71.
- [11] H.-F. Münzner, Isoparametrische Hyperflächen in Sphären II, *Mathematische Annalen*, 1981, **256**(2): 215-232.
- [12] H. Muto, The first eigenvalue of the Laplacian of an isoparametric minimal hypersurface in a unit sphere, *Mathematische Zeitschrift*, 1988, **197**(4): 531-549.
- [13] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian, *Duke Mathematical Journal*, 1981, **48**(4): 767-778.
- [14] H. Ozeki and M. Takeuchi, On some types of isoparametric hypersurfaces in spheres I, *Tôhoku Mathematical Journal, Second Series*, 1975, **27**(4): 515-559.

- [15] H. Ozeki and M. Takeuchi, On some types of isoparametric hypersurfaces in spheres II, Tôhoku Mathematical Journal, Second Series, 1976, **28**(1): 7-55.
- [16] R. Schoen, S.-T. Yau, Lectures on Differential Geometry, International Press, 1994.
- [17] B. Solomon, The harmonic analysis of cubic isoparametric minimal hypersurfaces I: Dimensions 3 and 6, American Journal of Mathematics, 1990, **112**(2): 157-203.
- [18] B. Solomon, The harmonic analysis of cubic isoparametric minimal hypersurfaces II: Dimensions 12 and 24, American Journal of Mathematics, 1990, **112**(2): 205-241.
- [19] T. Takahashi, Minimal immersions of Riemannian manifolds, Journal of the Mathematical Society of Japan, 1966, **18**(4): 380-385.
- [20] Z. Tang and W. Yan, Isoparametric foliation and Yau conjecture on the first eigenvalue, Journal of Differential Geometry, 2013, **94**(3): 521-540.
- [21] Z. Tang, Y. Xie and W. Yan, Isoparametric foliation and Yau conjecture on the first eigenvalue, II, Journal of Functional Analysis, 2014, **266**(10): 6174-6199.
- [22] S. T. Yau, Problem section, Seminar on differential geometry, Annals of Mathematics Studies, **102**, Princeton University Press, 1982.